

THE CONE PERCOLATION ON \mathbb{T}_d

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ABSTRACT. We study a rumour model from a percolation theory and branching process point of view. The existence of a giant component is related to the event where the rumour spreads out through an infinite number of individuals. We present sharp lower and upper bounds for the probability of that event, according to the distribution of the random variables that defines the radius of influence of each individual.

1. INTRODUCTION

We study long range dependent oriented percolation processes on a tree through its most basic propriety: the existence of a giant connected component. The starting point for approaches to rigorous percolation theory beyond the nearest neighbor independent setup on \mathbb{Z}^d is due to several authors around the nineties. Grimmett and Newman [5] in 1990 study percolation on $\mathbb{T}_d \times \mathbb{Z}$, Burton and Meester [3] in 1993 study phase transition for a long range independent percolation model on a stationary point process in \mathbb{R}^d , Lyons [8] put out the first version of his book *Probability on Trees* in 1994 while Benjamini and Schram [1] in 1996 have they *Percolation beyond \mathbb{Z}^d* published, just to name a few.

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Lebensztayn and Rodriguez [7] in 2008, propose a model on general graphs named *disk percolation* where a reaction chain starting from the origin of the graph, based on independent copies of a geometric random variables with parameter $q \in [0, 1]$, defines the existence or not of a giant component. They obtain a sufficient condition for the existence of phase transition based on q , which means the existence of a non-empty subcritical (no giant components) and supercritical (giant components with positive probability) phases. They associate their model to a rumour or an epidemic process. In this paper, instead of working in a general family of graphs we focus on homogeneous trees and instead of fixing the random variable which defines the radius of infection or the radius of influence of each vertex to be geometric, we consider general random variables. So, as a result, instead of having a phase transition phenomena depending on a point in a parametric space, we have that phenomena depending on the family of general positive random variables.

We consider a process which allows us to associate the activation dynamic on the set of vertices to a discrete rumour process. Individuals become spreaders as soon as they heard about the rumour. Next time, they propagate the rumour within their *radius of influence* and immediately become stiflers. Our main interest is to establish whether the process has positive probability of involving an infinite set of individual. Besides, we present sharp lower and upper bounds

for the probability of that event, according to the general distribution of the random variables that defines the *radius of influence* of each individual. We say that the process *survives* if the amount of vertices involved is infinite. Otherwise we say the process *dies out*.

Consider \mathbb{T}_d the homogeneous tree such that each vertex has $d + 1$ neighbours, $d \geq 2$. Let $\mathcal{V}(\mathbb{T}_d)$ the set of vertices of \mathbb{T}_d . We single out one vertex from $\mathcal{V}(\mathbb{T}_d)$ and call this \mathcal{O} , the origin. For each two vertices $u, v \in \mathcal{V}(\mathbb{T}_d)$, we say that $u \leq v$ if u belongs to the path connecting \mathcal{O} to v . Besides, for two vertices u, v such that $u \leq v$ let $d(u, v)$, be the distance between u and v , as the number of edges the path from u to v has. Now, let us define

$$\mathbb{T}_d^+(u) = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v\}.$$

Pick a $v \in \mathcal{V}(\mathbb{T}_d)$ such that $d(\mathcal{O}, v) = 1$ and consider $\mathbb{T}_d^+ = \mathbb{T}_d \setminus \mathbb{T}_d^+(v)$.

For \mathbb{T}_d^+ we define

$$\partial \mathbb{T}_d^+(u, n) = \{v \in \mathbb{T}_d^+ : d(u, v) = n\}.$$

Now we define the *Cone Percolation Process* in \mathbb{T}_d . Let $\{R_v\}_{v \in \mathcal{V}(\mathbb{T}_d)}$ and R be a set of independent and identically distributed random variables. We define $p_k = \mathbf{P}[R = k]$ for $k = 0, 1, \dots$. To avoid trivialities we assume $p_0 \in (0, 1)$. Besides, for each $u \in \mathcal{V}(\mathbb{T}_d)$, we define the random sets

$$B_u = \{v \in \mathcal{V}(\mathbb{T}_d) : d(u, v) \leq R_u\} \tag{1.1}$$

and consider the non-decreasing sequence of random sets $I_0 \subset I_1 \subset \dots$ defined as $I_0 = \{\mathcal{O}\}$ and inductively $I_{n+1} = \bigcup_{u \in I_n} B_u$ for all $n \geq 0$. Let $I = \bigcup_{n \geq 0} I_n$ be the connected component of the origin. Under the rumour process interpretation, I is the set of vertices which heard about the rumour. We say that the process survives if $|I| = \infty$, referring to the surviving event as V .

Consider \mathbb{P}_+ and \mathbb{P} the probability measures associated to the processes on \mathbb{T}_d^+ and \mathbb{T}_d (we do not mention the random variable R unless absolutely necessary). By a coupling argument one can see that for a fixed R

$$\mathbb{P}_+[V] \leq \mathbb{P}[V] \quad (1.2)$$

By the other side, by the definition of \mathbb{T}_d^+ and its relation with \mathbb{T}_d we have that for a fixed R

$$\mathbb{P}_+[V] = 0 \text{ if and only if } \mathbb{P}[V] = 0. \quad (1.3)$$

The paper is organized as follows. Section 2 presents the main results. Section 3 brings the proofs for the main results together with auxiliary lemmas and handy inequalities. Section 4 presents results for the heterogeneous setup of the *Cone Percolation Process*. Finally, in Section 5 we present examples where some conditions can be verified.

2. MAIN RESULTS

Theorem 2.1. *Consider the Cone Percolation Process on \mathbb{T}_d^+ with radius of influence R*

(I) *If $\mathbf{E}(d^R) > 1 + p_0$ then, $\mathbb{P}_+[V] > 0$,*

(II) *If $\mathbf{E}(d^R) \leq 2 - \frac{1}{d}$ then, $\mathbb{P}_+[V] = 0$.*

Let ρ and ψ be, respectively, the smallest non-negative root of the equations

$$\mathbf{E}(\rho^{d^R}) + (1 - \rho)p_0 = \rho, \quad (2.1)$$

$$\mathbf{E}(\psi^{\frac{d}{d-1}(d^R-1)}) = \psi. \quad (2.2)$$

Theorem 2.2. *Consider the Cone Percolation Process on \mathbb{T}_d^+ . Then,*

$$1 - \rho \leq \mathbb{P}_+(V) \leq 1 - \psi.$$

Theorem 2.3. *For the Cone Percolation Process on \mathbb{T}_d with radius of influence R , it holds that*

$$1 - \left(1 - \rho^{\frac{d+1}{d}}\right)p_0 - \mathbf{E}\left(\rho^{\frac{(d+1)}{d}d^R}\right) \leq \mathbb{P}[V] \leq 1 - \mathbf{E}\left(\psi^{\frac{(d+1)}{d-1}(d^R-1)}\right). \quad (2.3)$$

3. PROOFS

3.1. Auxiliary Processes. Let us define two auxiliary branching process, being the first one $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$. For this process, the associated random variable is X , assuming values in $\{0, d, d^2, \dots\}$ such that

$$\mathbf{P}[X = 0] = p_o,$$

$$\mathbf{P}[X = d^k] = p_k \text{ for } k = 1, 2, \dots$$

whose expectation is

$$\mathbf{E}[X] = \mathbf{E}[d^R] - p_0 \quad (3.1)$$

and whose generating function is

$$\varphi_X(s) = \mathbf{E}[s^X] = \mathbf{E}[s^{d^R}] + (1-s)p_0. \quad (3.2)$$

The second auxiliary process is $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$. For this process, the associated random variable is Y , assuming values in $\{0, d, d+d^2, \dots, \sum_{i=1}^k d^i\}$ such that

$$\mathbf{P}\left[Y = \frac{d(d^k - 1)}{d - 1}\right] = p_k \text{ for } k = 0, 1, 2, \dots$$

whose expectation is

$$\mathbf{E}[Y] = \frac{d}{d-1}(\mathbf{E}[d^R] - 1) \quad (3.3)$$

and whose generating function is

$$\varphi_Y(s) = \mathbf{E}[s^Y] = \mathbf{E}[s^{\frac{d}{d-1}(d^R-1)}]. \quad (3.4)$$

3.2. Proofs.

Proof of Theorem 2.1

By a coupling argument one can see that our process dominates $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$.

This process survives as long as $\mathbf{E}[X] > 1$ therefore from (3.1) our process survives if $\mathbf{E}[d^R] > 1 + p_0$, proving (I).

By the other side, also by a coupling argument, our process is dominated by $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$. That process dies out provided $\mathbf{E}[Y] \leq 1$ therefore from (3.3) our process dies out if $\mathbf{E}[d^R] \leq 2 - \frac{1}{d}$, proving (II). \square

Proof of Theorem 2.2

In order to find the extinction probability of $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ (Grimmett and Stirzaker([6, p.173])), let us consider the smallest non-negative root of the equation $\rho = \varphi_X(\rho)$. Therefore from (3.2)

$$\mathbf{E}[\rho^{d^R}] + (1 - \rho)p_0 = \rho$$

and by construction of the processes, as $\mathbb{P}_+[V^c] \leq \rho$, we have that

$$1 - \rho \leq \mathbb{P}_+[V].$$

In order to find the extinction probability of $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ (Grimmett and Stirzaker([6, p.173])), let us consider the smallest non-negative root of the equation $\psi = \varphi_Y(\psi)$. Therefore from (3.4)

$$\mathbf{E}[\psi^{\frac{d}{d-1}(d^R-1)}] = \psi$$

and by the construction of the processes, as $\mathbb{P}_+[V^c] \geq \psi$, we have that

$$\mathbb{P}_+[V] \leq 1 - \psi.$$

□

Proof of Theorem 2.3

Observe that except for the root, all vertices see towards infinity a tree like \mathbb{T}_d^+ . So, assuming $R_0 = k$ the probability for the process to survive is larger or equal than the probability of the process to survive from at least one of the $d^{k-1}(d+1)$ trees that have as root the furthest infected vertices. By the other side, still assuming $R_0 = k$, the probability for the process to survive in \mathbb{T}_d is smaller or equal than the probability for the process to survive from at least one of the

$(d+1)(d^k-1)(d-1)^{-1}$ trees like \mathbb{T}_d^+ that are seen from each active vertices by its own, independently. So, for $k = 1, 2, \dots$

$$1 - (1 - \mathbb{P}_+[V])^{(d+1)d^{k-1}} \leq \mathbb{P}[V|R_0 = k] \leq 1 - (1 - \mathbb{P}_+[V])^{\frac{(d+1)}{d-1}[d^k-1]}.$$

From this and from Theorem 2.2 follows (2.3). \square

4. HETEROGENEOUS CONE PERCOLATION ON \mathbb{T}_d^+

Suppose we have two sets of independent random variables, $\{R_z\}_{z \in \mathbb{N}}$ and $\{\bar{R}_v\}_{v \in \mathcal{V}(\mathbb{T}_d^+)}$, such that for all $z \in \mathbb{N}$ and all $u \in \mathcal{V}$ such that $d(\mathcal{O}, u) = z$, \bar{R}_u and R_z are equally distributed. Besides assume $\mathbb{P}[R_z = 0] < 1$ for all $z \in \mathbb{N}$.

We define the *Heterogeneous Cone Percolation Process* from the set of B_u defined in (1.1). For $n \in \mathbb{N}$ fixed and $u \leq v \in \mathcal{V}(\mathbb{T}_d^+)$, consider the event

$$V_{u,v}^n : \text{Process starting from } u \text{ reaches } v \text{ in at most } n \text{ steps.}$$

For a fixed integer n , let $X_0^n = \{\mathcal{O}\}$. Besides, for $j = 1, 2, \dots$ we define

$$X_j^n = \bigcup_{u \in X_{j-1}^n} \{v \in \partial T_n^u : V_{u,v}^n \text{ occurs}\}.$$

Again, for all $j = 1, 2, \dots$ consider

$$Z_j^n = |X_j^n|.$$

So, for all fixed positive integer n , $\{Z_j^n\}_{j \geq 0}$ is a branching process dominated by the number of vertices $v \in \partial T_{jn}^{\mathcal{O}}$ which are activated.

Lemma 4.1. *Consider n fixed. For $\mu_j := \mathbf{E}[Z_j^n]$, the mean number of offspring on generation j for the process $\{Z_j^n\}_{j \geq 0}$, it holds that*

$$\mu_j = d^n \rho_j^n,$$

where $\rho_j^n = \mathbb{P}[V_{u,v}^n]$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

Proof of Lemma 4.1

For fixed j and n , consider $\partial T_n^v = \{u_1, u_2, \dots, u_{d^n}\}$. So we can write Z_j^n as $\sum_{i=1}^{d^n} I_{\{V_{v,u_i}^n\}}$. Taking expectation in both sides finishes the proof.

Lemma 4.2. *Consider n fixed and $\rho_j^n = \mathbb{P}[V_{u,v}^n]$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.*

$$\rho_j^n \geq \prod_{k=0}^{n-1} [1 - \prod_{i=0}^k \mathbb{P}[R_{jk+i} < k+1-i]].$$

Proof of Lemma 4.2

For any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$ we have that

$$V_{u,v}^n = \bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^k \{R_{jn+i} \geq k+1-i\} \right]$$

and so

$$\begin{aligned} \rho_j^n &= \mathbb{P} \left(\bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^k \{R_{jn+i} \geq k+1-i\} \right] \right) \\ &\geq \prod_{k=0}^{n-1} \mathbb{P} \left(\bigcup_{i=0}^k \{R_{jn+i} \geq k+1-i\} \right). \end{aligned}$$

The inequality is a consequence of the FKG inequality (Grimmett [4, p.34]). \square

Theorem 4.3. *The Heterogeneous Cone Percolation Process in \mathbb{T}_d^+ has a giant component with positive probability if for some fixed n ,*

$$\liminf_{j \rightarrow \infty} d^n \prod_{k=0}^{n-1} [1 - \prod_{i=0}^k \mathbb{P}[R_{jk+i} < k+1-i]] > 1. \quad (4.1)$$

Proof of Theorem 4.3

From Theorem 1 in Souza & Biggins ([2, p.39]), a branching process in varying environments is uniformly supercritical if

$$\liminf_{j \rightarrow \infty} \mu_j > 1.$$

From Lemma 4.1 and Lemma 4.2, that is what happens if (4.1) holds.

From the fact that

$$\frac{Z_j^n}{\mathbf{E}[Z_j^n]} \leq \frac{1}{\rho_j^n}$$

one can see that the *Heterogeneous Cone Percolation Process* has a giant component with positive probability if

$$\liminf_{j \rightarrow \infty} d^n \prod_{k=0}^{n-1} [1 - \prod_{i=0}^k \mathbb{P}[R_{jk+i} < k+1-i]] > 1.$$

\square

5. EXAMPLES

Example 5.1. Consider a *Cone Percolation Process* in \mathbb{T}_d , assuming

$$\mathbf{P}[R = 1] = p = 1 - \mathbf{P}[R = 0].$$

In words $R \sim \mathcal{B}(p)$.

- If $p > d^{-1}$ then, $\mathbb{P}[V] > 0$.
- If $p \leq d^{-1}$ then, $\mathbb{P}[V] = 0$.

By the definition, one can see that

$$\mathbb{P}[V^c] = (1 - p) + (\mathbb{P}_+[V^c])^{d+1}p.$$

Observing that the upper and lower process presented by $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ presented in session 3.1 are the same, we see that

$$\mathbb{P}[V] = p(1 - \psi^{d+1}),$$

being ψ the solution of

$$p\psi^d - \psi + 1 - p = 0.$$

Example 5.2. Consider a *Cone Percolation Process* in \mathbb{T}_d , assuming

$$\mathbf{P}(R = k) = (1 - p)p^k, k = 0, 1, 2, \dots$$

In other words $R \sim \mathcal{G}(1 - p)$. From Theorem 2.3

- If $dp^2 - 2dp + 1 < 0$ then, $\mathbb{P}[V] > 0$.
- If $2pd \leq 1$ then, $\mathbb{P}[V] = 0$.

As a consequence of this and (1.3), for d fixed

$$\frac{1}{2d} < \inf\{p : \mathbb{P}[V] > 0\} \leq 1 - \sqrt{1 - \frac{1}{d}}.$$

Example 5.3. Consider a *Cone Percolation Process* in \mathbb{T}_d , assuming

$$\mathbf{P}(R = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n$$

- If $(pd + 1 - p)^n - (1 - p)^n > 1$ then, $\mathbb{P}[V] > 0$.

- If $2d - d(pd + 1 - p)^n \geq 1$ then, $\mathbb{P}[V] = 0$.

Let $d = 2$ and $R \sim \mathcal{B}(4, \frac{1}{2})$.

Therefore ρ and ψ are, respectively, solutions of

$$x^{16} + 4x^8 + 6x^4 + 4x^2 - 16x + 1 = 0,$$

$$x^{30} + 4x^{14} + 6x^6 + 4x^2 - 16x + 1 = 0.$$

So $\rho = 0.0635146$ and $\psi = 0.06350850$, which implies that

$$0.937435919 \leq \mathbb{P}[V] \leq 0.937435962.$$

Let $d = 4$ and $R \sim \mathcal{B}(4, \frac{1}{4})$.

Therefore ρ and ψ are, respectively, solutions of

$$x^{256} + 12x^{64} + 54x^{16} + 108x^4 - 256x + 81 = 0,$$

$$x^{340} + 12x^{84} + 54x^{20} + 108x^4 - 256x + 81 = 0.$$

So $\rho = 0.3208787235$ and $\psi = 0.3208787200$, which implies that

$$0.682158629 \leq \mathbb{P}[V] \leq 0.682158630.$$

Example 5.4. Consider a *Heterogeneous Cone Percolation Process* on

\mathbb{T}_d^+ , assuming that R_j are Bernoullis, that is,

$$\mathbb{P}[R_j = 1] = 1 - \mathbb{P}[R_j = 0] \text{ for } j = 0, 1, 2, \dots$$

By applying Theorem 4.3 with $n = 1$ one can see that the *Heterogeneous Cone Percolation Process* on \mathbb{T}_d^+ survives with positive probability if

$$\liminf_{j \rightarrow \infty} d\mathbb{P}[R_j = 1] > 1.$$

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